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Pfaffian solutions for the Manin-Radul-Mathieu SUSY KdV and SUSY sine-Gordon equations

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Abstract

We reduce the vectorial binary Darboux transformation for the Manin-Radul supersymmetric KdV system in such a way that it preserves the Manin-Radul-Mathieu supersymmetric KdV equation reduction. Expressions in terms of bosonic Pfaffians are provided for transformed solutions and wave functions. We also consider the implications of these results for the supersymmetric sine-Gordon equation. © 1998 Elsevier Science B.V. All rights reserved.

1. Introduction

The supersymmetric version of the Korteweg-de Vries (KdV) system was introduced by Manin and Radul in [9]. Thereafter many integrable equations have been extended in this way. The role of the KdV equation in two dimensional quantum gravity lead the group of Alvarez-Gaumé to search for analogous structures for supersymmetric two dimensional quantum gravity [1]. Their results indicated that the supersymmetric extensions of the KdV equation might be relevant in the study of SUSY 2d quantum gravity.

The Manin-Radul SUSY KdV has a distinguished reduction [9,8], a single equation that we shall call Manin-Radul-Mathieu SUSY KdV. This equation is closely related to the SUSY sine-Gordon equation [2]. In this note we extend to these equations the Darboux transformations, providing in this manner efficient ways to construct explicit solutions of these two equations. Our scheme is a suitable reduction of the one proposed in [3,4] (see also [5] for super Wroński determinant soutions), that leads us to express the new solutions in terms of bosonic Pfaffians.

The layout of the paper is as follows. First, in §2, we recall the reader the basic facts regarding the vectorial binary Darboux transformation for the Manin-Radul SUSY KdV system, then, in §3, we reduce these results to the Manin-Radul-Mathieu SUSY KdV equation. Finally, in §4, we conclude the letter by applying these techniques to the SUSY sine-Gordon equation. Let us remark that for each

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case we give explicit examples, namely a supersoliton for the Manin-Radul-Mathieu SUSY KdV and a superkink solution of the SUSY sine-Gordon equation —these examples are obtained by dressing the zero solution. Obviously, multi-supersoliton and multi-superkink solutions are immediately constructed within our scheme using the mentioned bosonic Pfaffians.

2. Vectorial binary Darboux transformations for the Manin-Radul supersymmetric KdV system

The MR SUSY KdV system is defined in terms of three independent variables ϑ, x, t, where $\vartheta \in \mathbb{C}_a$ is an odd supernumber, and $x,t \in \mathbb{C}_c$ are even supernumbers, and two dependent variables $\alpha(\vartheta,x,t), u(\vartheta,x,t)$, where α is an odd function taking values in $\mathbb{C}_a$ and u is even function with values in $\mathbb{C}_c$. A basic ingredient is a superderivation defined by $D := \partial_\vartheta + \vartheta\partial_x$. The system is

$$\alpha_t = \frac{1}{4}\left(\alpha_{xxx} + 3(\alpha D\alpha)_x + 6(\alpha u)_x\right),$$

$$u_t = \frac{1}{4}\left(u_{xxx} + 6uu_x + 3\alpha_x Du + 3\alpha(Du_x)\right), \quad (1)$$

where we use the notation $f_x := \partial f/\partial x$ and $f_t := \partial f/\partial t$.

Let E be a supervector space over $\boldsymbol{\Lambda} := \mathbb{C}_a \oplus \mathbb{C}_c$ and ℓ an even linear operator over E; then, the linear system

$$\psi_{xx} + \alpha D\psi + u\psi - \ell\psi = 0,$$

$$\psi_t - \frac{1}{2}\alpha(D\psi_x) - \ell\psi_x - \frac{1}{2}u\psi_x + \frac{1}{4}\alpha_x D\psi + \frac{1}{4}u_x\psi = 0, \quad (2)$$

has as its compatibility condition the MR SUSY KdV system (1). Eqs. (1) are also the compatibility condition of adjoint linear system:

$$\psi^*_{xx} + D(\alpha\psi^*) + u\psi^* - \psi^* m = 0,$$

$$\psi^*_t + \frac{1}{2}\alpha D\psi^*_x - \psi^*_x m - \frac{1}{2}(u + D\alpha)\psi^*_x + \frac{1}{4}D(\alpha_x\psi^*) + \frac{1}{4}u_x\psi^* = 0, \quad (3)$$

where $\psi^*(\vartheta,x,t) \in \tilde{E}^*$ is a linear function on the supervector space $\tilde{E}$, and m is an even linear operator.

In order to construct Darboux transformations for these linear systems we need to introduce a linear operator, say $\Omega[\psi,\psi^*]: \tilde{E} \to E$, bilinear in ψ and ψ^*, defined by the compatible equations

$$D\Omega[\psi,\psi^*] = \psi \otimes \psi^*,$$

$$\begin{aligned}\Omega[\psi,\psi^*]_t = {} & \ell\Omega[\psi,\psi^*]_x + \Omega[\psi,\psi^*]_x m \\ & - D\left(\psi_x \otimes \psi^*_x + \frac{1}{2}uD\Omega[\psi,\psi^*]\right) \\ & - \frac{1}{4}\alpha_x D\Omega[\psi,\psi^*] \\ & - \frac{1}{2}(D\psi) \otimes ((D\alpha)\psi^* - \alpha(D\psi^*)) \\ & + \frac{1}{2}\alpha(\psi \otimes \psi^*_x - \psi_x \otimes \psi^*) \end{aligned} \quad (4)$$

such that

$$\ell\Omega[\psi,\psi^*] - \Omega[\psi,\psi^*]m = D(\psi_x \otimes \psi^* - \psi \otimes \psi^*_x) - \alpha\psi \otimes \psi^*. \quad (5)$$

Now we are ready to present the following:

The vectorial binary Darboux transformation [4,6]. Let $\xi(\vartheta,x,t)$ be an even vector of the supervector space V, of total dimension N, satisfying Eq. (2), $\xi^*(\vartheta,x,t)$ an odd linear functional of the dual supervector space V^* solving Eq. (3) and $\Omega[\xi,\xi^*]$ a non singular even linear operator on V, $\det\Omega[\xi,\xi^*]_{\text{body}} \neq 0$, defined in terms of the compatible Eqs. (4) and (5). Then, the objects

$$\hat{\psi} := \psi - \Omega[\psi,\xi^*]\Omega[\xi,\xi^*]^{-1}\xi,$$

$$\hat{\psi^*} := \psi^* - \xi^*\Omega[\xi,\xi^*]^{-1}\Omega[\xi,\psi^*],$$

$$\hat{\alpha} = \alpha - 2D^3 \ln\det\Omega[\xi,\xi^*],$$

$$\hat{u} = u + 2\hat{\alpha}D\ln\det\Omega[\xi,\xi^*] + 2\left(\frac{\sum_{j=1}^{N} D(\xi)_j \det\Omega[\xi,\xi^*]_j}{\det\Omega[\xi,\xi^*]}\right)_x,$$

where $\Omega[\xi,\xi^*]_j$ is an operator with associated supermatrix obtained from the corresponding one of $\Omega[\xi,\xi^*]$ by replacing the j-th column by ξ, satisfy the Eqs. (2) and (3) whenever the unhatted variables do. Thus, $\hat{\alpha}$ and $\hat{u}$ are new solutions of (1).

Observe that here we are using ordinary determinants, this is possible because we are considering only even matrices, so that its coefficients conmute, and also because we have finite total dimension N.

3. The reduction to the Manin-Radul-Mathieu supersymmetric KdV equation

The equations (1) admits the reduction, $u=0$, due to Manin and Radul [9] and studied later by Mathieu [8]. The system reduces then to the single equation

$$4\alpha_t = \alpha_{xxx} + 3(\alpha D\alpha)_x, \tag{6}$$

the so called Manin-Radul-Mathieu supersymmetric KdV equation.

Obviously this equation is the compatibility condition of the following linear system

$$\psi_{xx} + \alpha D\psi - \ell\psi = 0,$$
$$\psi_t - \frac{1}{2}\alpha(D\psi_x) - \ell\psi_x + \frac{1}{4}\alpha_x D\psi = 0. \tag{7}$$

We first observe that given an even solution ξ of (2) then $D\xi^{\mathrm{t}}$ is an odd solution of (3) if and only if $Du=0$. Second, if $u=u_0$, a constant, given ξ and ψ solutions of (2) we can take $\xi^* = D\xi^{\mathrm{t}}$ and $\psi^* = D\psi^{\mathrm{t}}$ as solutions of (3) whenever $m=\ell^{\mathrm{t}}$, then

$$\Omega[\psi,\xi^*] + \Omega[\xi,\psi^*]^{\mathrm{t}} - \psi\otimes\xi^t$$

is a constant linear operator.

Now, if we perform the vectorial binary Darboux transformation for the MR SUSY KdV system induced by the above transformation data and impose

$$\Omega[\xi,\xi^*] + \Omega[\xi,\xi^*]^{\mathrm{t}} = \xi\otimes\xi^t,$$
$$\Omega[\psi,\xi^*] + \Omega[\xi,\psi^*]^{\mathrm{t}} = \psi\otimes\xi^t, \tag{8}$$

then, the transformed wave functions $\hat{\psi}$ and $\hat{\psi}^*$ are linked by $\hat{\psi}^* = D\hat{\psi}^{\mathrm{t}}$, so that $D\hat{u}=0$. Moreover, one can show that $\hat{u}=u_0$, so if $u=0$ then $\hat{u}=0$.

We can summarize these results in the following

Proposition. Given solutions $\psi\in\mathbb{C}_{\mathrm{c}}$ and ξ, an even vector in V, of (7) and potentials subject to (8); then, the vectorial binary Darboux transformation preserves the Manin-Radul-Mathieu supersymmetric KdV equation reduction.

At this point one could apply the Grammian type expressions of the vectorial binary Darboux transformation of the MR SUSY KdV system to get new solutions of the MRM SUSY KdV equation from given ones. However, these expressions for the components of the wave functions and fields can be rewritten compactly in terms of ordinary Pfaffians, the derivation of this results follows the lines given in [7]. But first let us remind the reader what is a Pfaffian of an even-dimensional skew-symmetric matrix. If $S=(s_{ij})_{i,j=1,\dots,2M}$ is an even-dimensional skew matrix, $s_{ij}+s_{ji}=0$, its Pfaffian is defined as

$$\operatorname{Pf} S = \sum_{\pi\in P}\operatorname{sgn}(\pi)\prod_{k=0}^{M-1} s_{\pi(2k+1)\pi(2k+2)}$$

where P is the set of permutations π of the $2M$ first natural numbers such that

1. For $i,j=0,\dots,M-1$, with $i<j$, then $\pi(2i+1) < \pi(2j+1)$.
2. For any $j=1,\dots,M$ one has $\pi(2j-1)<\pi(2j)$.

For example, for $M=2$ we have $\operatorname{Pf} S = s_{12}s_{34} - s_{13}s_{24} + s_{14}s_{23}$. An important property of Pfaffians is that for any matrix M we have $\operatorname{Pf}(M^{\mathrm{t}}SM) = \det(M)\operatorname{Pf} S$, from where it follows that $\det S = (\operatorname{Pf} S)^2$. Obviously, as we already did with ordinary determinants, we can extend this construction to have Pfaffians of even linear operators on supervector spaces with even total dimension, $N=2M$.

Notice that the potential matrices $\Omega[\xi,\xi^*]$ and $\Omega[\psi,\xi^*]$ split as follows

$$\Omega[\xi,\xi^*] = \frac{1}{2}\left[\xi\otimes\xi^{\mathrm{t}} + S\right], \tag{9}$$

$$\Omega[\psi,\xi^*] = \frac{1}{2}\left[\psi\xi^{\mathrm{t}} + S[\psi,\xi]^{\mathrm{t}}\right], \tag{10}$$

where S is skew-symmetric and

$$DS = \xi\otimes D\xi^{\mathrm{t}} - (D\xi)\otimes\xi^{\mathrm{t}},$$
$$DS[\psi,\xi] = \psi D\xi - (D\psi)\xi.$$

In terms of S the constraint (5) reads

$$\ell S - S\ell^{t} = 2\big[(D\xi_x) \otimes (D\xi^{t}) - (D\xi) \otimes \big(D\xi_x^{t}\big) + \xi_x \otimes \xi_x^{t} \big] - \ell\xi \otimes \xi^{t} - \xi \otimes \xi^{t}\ell^{t}. \qquad (11)$$

Pfaffian form of the reduced transformation. The reduction of the Grammian determinant type solutions of the MR SUSY KdV system to the MRM SUSY KdV equation gives

1. For N even

$$\hat{\alpha} = \alpha - 4D^3(\ln \mathrm{Pf}(S)),$$

$$\hat{\psi} = \frac{\mathrm{Pf}\begin{pmatrix} 0 & \psi & S[\psi,\xi]^{t} \\ -\psi & 0 & -\xi^{t} \\ -S[\psi,\xi] & \xi & S \end{pmatrix}}{\mathrm{Pf}(S)}.$$

2. For N odd

$$\hat{\alpha} = \alpha - 4D^3\left(\ln \mathrm{Pf}\begin{pmatrix} 0 & -\xi^{t} \\ \xi & S \end{pmatrix}\right),$$

$$\hat{\psi} = \frac{\mathrm{Pf}\begin{pmatrix} 0 & S[\psi,\xi]^{t} \\ -S[\psi,\xi] & S \end{pmatrix}}{\mathrm{Pf}\begin{pmatrix} 0 & -\xi^{t} \\ \xi & S \end{pmatrix}}.$$

If $\ell = \mathrm{diag}(\ell_1,\dots,\ell_N)$ is a diagonal matrix with $\ell_i \in \mathbb{C}_{\mathrm{c}}$ all different, $\ell_i \neq \ell_j$ for $i \neq j$, then (11) imply that the components of ξ, ξ_i, $i = 1,\dots,N$ are subject to

$$2(D\xi_i)_x(D\xi_i) + (\xi_i)_x^2 - \ell_i \xi_i^2 = 0 \qquad (12)$$

and determines $S = (s_{ij})$ as

$$s_{ij} = \frac{2}{\ell_i - \ell_j}\big[(D\xi_i)_x(D\xi_j) - (D\xi_i)(D\xi_j)_x + (\xi_i)_x(\xi_j)_x - (\ell_i + \ell_j)\xi_i\xi_j \big].$$

In particular, for zero background $\alpha = 0$,

$$\xi_i(\theta,x,t) = (a_{i,0}^{+} + \theta a_{i,1}^{+})\exp\big(k_i x + k_i^3 t\big) + (a_{i,0}^{-} + \theta a_{i,1}^{-})\exp\big(-k_i x - k_i^3 t\big),$$

where $k_i^2 = \ell_i$, $a_{i,0}^{\pm}$ and $a_{i,1}^{\pm}$ even and odd supernumbers, respectively, and (12) simply is

$$a_{i,1}^{+} a_{i,1}^{-} = k_i a_{i,0}^{+} a_{i,0}^{-}.$$

For example, the solution for $N = 1$ is [6]

$$\hat{\alpha} = \alpha - 4\frac{(D\xi_x)\xi - \xi_x(D\xi)}{\xi^2},$$

that for $\alpha = 0$ gives

$$\hat{\alpha} = -\frac{8k(a_1^{+}a_0^{-} - a_1^{-}a_0^{+})}{\big(a_0^{+}\exp(\eta) + a_0^{-}\exp(-\eta)\big)^2} \times \left[1 - 2\theta\frac{a_1^{+}\exp(\eta) + a_1^{-}\exp(-\eta)}{a_0^{+}\exp(\eta) + a_0^{-}\exp(-\eta)}\right],$$

where $\eta(x,t) := kx + k^3 t$ and we are assuming that the even quantity $a_0^{+}\exp(\eta) + a_0^{-}\exp(-\eta)$ is invertible. Which can be considered as the typical 1-soliton of the MRM SUSY KdV equation; hence, our Pfaffian solutions, when applied to the zero background, provides the multi-soliton solutions of this equation.

4. Pfaffian solutions of the supersymmetric sine-Gordon equation

The SUSY sine-Gordon equation [2] is

$$DD_t\Phi = \sin(\Phi) \qquad (13)$$

where $D_t = \frac{\partial}{\partial\theta_t} + \theta_t\frac{\partial}{\partial t}$ and now $\theta,\theta_t \in \mathbb{R}_{\mathrm{a}}$ and $\Phi, x, t \in \mathbb{R}_{\mathrm{c}}$. The linear system

$$\psi_x = \mathrm{i}(D\Phi)D\psi + K\bar{\psi}$$

$$D_t\psi = -\frac{1}{2}K^{-1}\exp(\mathrm{i}\Phi)D\bar{\psi}, \qquad (14)$$

has as their compatibility (13), here ψ is an even vector in a supervector space and K an invertible real even operator over it. It can be easily shown that

$$\psi_{xx} + \alpha D\psi - \ell\psi = 0$$

where $\alpha := \gamma D\gamma - \mathrm{i}\gamma_x$ with $\gamma = D\Phi$ an odd real supervariable and $\ell = K^2$.

A standard application [11,10] of the results of the previous section yields

Theorem. Let ξ be an even vector solution in the supervector space V, of total dimension N, of (14) so that (11) are satisfied; then,

$$\hat{\Phi} - \Phi = \begin{cases} 4\arg \mathrm{Pf}\, S, & \text{when N is even}, \\ 4\arg \mathrm{Pf} \begin{pmatrix} 0 & -\xi^{\mathrm{t}} \\ \xi & S \end{pmatrix}, & \text{when N is odd}. \end{cases}$$

is a new solution of (13).

When $K = \mathrm{diag}(k_1, \dots, k_N)$ and $\Phi = 0$ the solution of (14) has components

$$\xi_i = \left[\left(1 - \frac{1}{2}\theta_t \theta \right) A^+_{i,0} + \left(\theta - \frac{1}{2k_i}\theta_t \right) A^+_{i,1} \right] \times \exp(\eta_i(x,t)) + \mathrm{i} \left[\left(1 - \frac{1}{2}\theta_t \theta \right) A^-_{i,0} + \left(\theta + \frac{1}{2k_i}\theta_t \right) A^-_{i,1} \right] \times \exp(-\eta_i(x,t))$$

where now

$$\eta_i(x,t) := k_i x - \frac{1}{4k_i} t, \qquad A^+_{i,1} A^-_{i,1} = k_i A^+_{i,0} A^-_{i,0}.$$

For $N = 1$ we get the superkink solution

$$\hat{\Phi} = 4\arctan\left(\frac{\left(1 - \frac{1}{2}\theta_t \theta \right) A^-_0 + \left(\theta + \frac{1}{2k}\theta_t \right) A^-_1}{\left(1 - \frac{1}{2}\theta_t \theta \right) A^+_0 + \left(\theta - \frac{1}{2k}\theta_t \right) A^+_1} \times \exp\left(-2kx + \frac{1}{2k}t \right) \right),$$

if we assume that $A^+_0 \in \mathbb{R}_c$ is invertible we can write

$$\Phi = 4\arctan\left(\frac{1}{A^+_0} \left(A^-_0 + \theta \frac{A^-_1 A^+_1 - A^+_1 A^+_0}{A^+_0} - \theta_t \frac{A^-_1 A^+_1 + A^+_1 A^+_0}{2kA^+_0} - \theta_t \theta A^-_0 \right) \times \exp\left(-2kx + \frac{1}{2k}t \right) \right).$$

The formulae of our theorem obviously contains the supersymmetric extension of topological solitons like the multi-kinks solutions.

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